Assignment 11

Hand in no. 1, 2, 3 and 4 by Nov 30.

1. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $\mathbb{R}$ (you may draw a table):
   (a) $A = \{n/2^m : \ n, m \in \mathbb{Z}\}$,
   (b) B, all irrational numbers,
   (c) $C = \{0, 1, 1/2, 1/3, \cdots\}$,
   (d) $D = \{1, 1/2, 1/3, \cdots\}$,
   (e) $E = \{x : \ x^2 + 3x - 6 = 0 \}$,
   (f) $F = \cup_k (k, k+1), k \in \mathbb{N}$,
2. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $C[0, 1]$ (you may draw a table):
   (a) $\mathcal{A}$, all polynomials whose coefficients are rational numbers,
   (b) $\mathcal{B}$, all polynomials,
   (c) $\mathcal{C} = \{f : \ \int_0^1 f(x)dx \neq 0\}$,
   (d) $\mathcal{D} = \{f : \ f(1/2) = 1 \}$.
3. Use Baire Category Theorem to show that transcendental numbers are dense in the set of real numbers.
4. A set E in a metric space is called a perfect set if, for each point $x \in E$ and $r > 0$, the ball $B_r(x) \bigcap E$ contains a point different from x.
   (a) For each x in the perfect set E, there exists a sequence in E consisting of infinitely many distinct points converging to x.
   (b) Every complete perfect set is uncountable. Hint: Use Baire Category Theorem.
   (c) Is (b) true without completeness?
5. Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$.
   (a) Show that $\|x\| \leq C\|x\|_2$ for some C where $\| \cdot \|_2$ is the Euclidean metric.
   (b) Deduce from (a) that the function $x \mapsto \|x\|$ is continuous with respect to the Euclidean metric.
   (c) Show that the inequality $\|x\|_2 \leq C'\|x\|$ for some C' also holds. Hint: Observe that $x \mapsto \|x\|$ is positive on the unit sphere $\{x \in \mathbb{R}^n : \ \|x\|_2 = 1\}$ which is compact.
   (d) Establish the theorem asserting any two norms in a finite dimensional vector space are equivalent.
6. Let $\mathcal{F}$ be a subset of $C(X)$ where X is a complete metric space. Suppose that for each $x \in X$, there exists a constant M depending on x such that $|f(x)| \leq M, \ \forall f \in \mathcal{F}$. Prove that there exists an open set G in X and a constant C such that $\sup_{x \in G} |f(x)| \leq C$ for all $f \in \mathcal{F}$. Suggestion: Consider the decomposition of X into the sets $X_n = \{x \in X : |f(x)| \leq n, \ \ \forall f \in \mathcal{F}\}$.
7. Optional. A function is called non-monotonic if if is not monotonic on every subinterval. Show that all non-monotonic functions form a dense set in $C[a, b]$. Hint: Consider the sets
$$\mathcal{E}_n = \{f \in C[a, b] : \exists x \text{ such that } (f(y) - f(x))(y - x) \geq 0, \ \forall y, \ |y - x| \leq 1/n\}.$$